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On wave scattering by random inhomogeneities, with application to the theory of weak bores

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This paper considers the propagation and scattering of waves in dispersive and non-dispersive media containing random inhomogeneities. A detailed discussion of the propagation of the *coherent* component of the wave field is presented, and a rather general method for obtaining the absorption coefficient and scattering cross-section per unit volume of medium is described. A major objective is to derive *macroscopic* equations for the propagation of the coherent field in the sense that the damping due to scattering appears in the wave equation as *pseudo-viscosity*. The paper is concluded by a detailed application of the theory to the problem of the propagation of surface gravity waves over a stretch of shallow water with a 'rough' bed. By including finite amplitude effects a balance is achieved between the dispersion of the pseudo-viscous term and the non-linear convective terms, which enables the steady profile of a weak bore to be calculated.

1. Introduction

It is known that when small inhomogeneities are present in a wave-bearing medium the effect is to produce a gradual randomization of an initially coherent wave field. The process may be imagined as one in which wave energy is continuously scattered out of the coherent signal and, since the state of the scatterers is random, the ultimate effect is one in which an assembly (or 'gas') of random wave packets is generated, its creation being at the expense of the energy of the coherent wave field. Hence, if attention be confined solely to the propagation of the coherent field, in ignorance, so to speak, of the existence of the accompanying random wave packet 'gas', one would expect to obtain a *macroscopic* picture in which the process of decay would be represented by the appearance of a *pseudo-viscous* term in an equation describing the propagation of the coherent field.

Such mean field equations have been derived by Keller (1964) and Karal & Keller (1964), and applied to the study of elastic and electromagnetic waves. An alternative treatment of such problems has been discussed by Howe (1971). Howe indicates how the mean field equation may be derived to any desired degree of approximation by first calculating the random field from an equation describing a multiple collision process. He considers in detail the lowest-order approximation (the *binary collision approximation*) to the equation describing the propagation of coherent transverse waves along an infinite stretched string, whose density is a random function of position. By deriving the equation describing the relation between the wave-number and frequency of infinitesimal sinusoidal wave trains (the *dispersion relation*), he was able to show that at all frequencies the random fluctuations in density cause the mean field to be damped. In the two limiting cases of wavelength large and small compared to the correlation scale of the random fluctuations the dispersion relation could be reduced to a simplified form. Corresponding approximate differential equations for the coherent field, containing 'viscous' damping terms, were then derived in much the same manner that one derives Schrödinger's equation from the Hamilton–Jacobi equation of classical mechanics.

The methods employed in these earlier papers tended to rely heavily on a knowledge of the Green's function of the homogeneous problem. A principal aim of the present paper (§3) is to show how valuable information concerning the binary collision approximation to the coherent field equation may be derived without a knowledge of the Green's function, or in cases where the form of the Green's function would lead to analytical difficulties. The procedure is particularly valuable when it is not possible to solve the homogeneous problem explicitly. Further, in the present paper we make no assumptions regarding the form of the correlation function of the random inhomogeneities, apart from supposing that such inhomogeneities are *stationary random functions*. We are thus able to derive a rather general expression for the decay rate of a coherent field wave packet of given wave-number and frequency, and also an expression for the total scattering cross-section per unit volume of the medium (§4).

Finally, the theory is illustrated in detail by consideration of the problem of the propagation of surface waves over a stretch of shallow water with a rough bed (§§5, 6). Actually, it is also possible to indicate here a further extension of the present theory to the case of weakly non-linear systems. Thus, the problem of the shallow water bore is treated from the point of view that the steady form of the bore is maintained by a balance between the non-linear convective terms in the shallow water equations and the dissipative term due to scattering off random fluctuations of the depth (§7). Such a theory also has application to the propagation of sonic booms through atmospheric turbulence (Ffowcs Williams & Howe 1971), as well as possibly to the intriguing problem of 'collisionless' shocks. The steady state of the Earth's bow shock wave (Hess 1968) might possibly be accounted for by the competition between non-linear convective terms and scattering due to plasma turbulence.

We begin by recalling the form of the coherent field equation.

2. The equation for the coherent field

Consider a wave field ϕ propagating in a dispersive or non-dispersive, non-random medium, and satisfying the equation,

$$L\phi = 0, \tag{2.1}$$

where L is a linear wave operator. When small random inhomogeneities are present in the medium let (2.1) take the modified form,

$$L\phi = G\phi, \tag{2.2}$$

where G is a random linear operator, which, for simplicity of exposition, we shall assume to have zero mean.

We now decompose the wave field in the random medium into two components $\overline{\phi}$ and ϕ', such that

$$\phi = \overline{\phi} + \phi'. \tag{2.3}$$

In (2.3) $\overline{\phi}$ represents the mean, or coherent, component of the field in the sense of an ensemble average. Then ϕ' represents the fluctuations of the actual field about this mean in any particular experimental realization. A fuller discussion of these points is given by Howe (1971).

The equation governing the propagation of the coherent field may be derived in several ways. The first rigorous derivation appears to be due to Keller (1964) and Karal & Keller (1964). An alternative derivation has also been discussed by the author (Howe 1971), and we shall quote the result presented there.

Now, the random operator G on the right of (2.2) is assumed to have zero mean, i.e. if an over-bar denotes the ensemble average, then $\overline{G} \equiv 0$. However, the ensemble average of a quantity such as G^2 does not necessarily vanish. For simplicity, however, let us define the operator $\overline{G}$ by

$$\overline{G}\,.\,\psi = \overline{G\psi}, \tag{2.4}$$

where ψ is an arbitrary random or non-random function or operator. Then (Howe 1971, equation (5.11)) the equation for the coherent field may be shown to have the form,

$$L\overline{\phi} = \overline{G} \sum_{n=0}^{\infty} \{L^{-1}G - L^{-1}\overline{G}\}^n \, L^{-1}G\overline{\phi}, \tag{2.5}$$

where L^{-1} denotes the Green's function operator inverse to L.

Equation (2.5) governs the evolution of the coherent waves $\overline{\phi}$ alone, without specific reference to the random field ϕ'. It is to be solved as an initial value problem, the form of the coherent wave $\overline{\phi}$ being specified at some initial instant. The equation is derived on the assumption that the random field ϕ' is generated solely by scattering of energy out of the coherent field by the random inhomogeneities, so that initially the random field is null.

3. The theory of binary collision scattering

The zeroth-order term, $\overline{GL^{-1}G}\,.\,\overline{\phi}$, on the right of the mean field equation (2.5), is quadratic in the random fluctuations of the medium. It is often adequate to neglect higher-order terms and adopt the *binary collision approximation*,

$$L\overline{\phi} = \overline{GL^{-1}G}\,.\,\overline{\phi}, \tag{3.1}$$

to the mean field equation. In the first instance, higher-order collision terms involve the cube and higher powers of the random inhomogeneity. On the other hand, however, the result of the operation of G on $\overline{\phi}$ is not necessarily small, so that in using (3.1), apart from requiring that the random fluctuations in the medium be small, it is also necessary to suppose that $\bar{\phi}$ satisfies certain *smoothness* conditions associated with the random operator G. Provided that such a smoothness condition can be satisfied, it is interesting to note that, if the fluctuations in G have a symmetric distribution, then the tertiary collision term in (2.5), corresponding to $n = 1$, automatically vanishes, and (3.1) is then valid up to and including third-order in the random fluctuations.

In § 3 we present a somewhat general analysis of the binary collision equation (3.1). This is certainly a valid approximation for the treatment of a given mean field wave packet of fixed wave-number, provided that the random fluctuations of the medium are sufficiently small.

Now (3.1) must be solved as an initial value problem, the form of the mean field $\bar{\phi}$ being specified at some initial instant of time. It is therefore equivalent to solving the pair of scattering equations,

$$\left.\begin{aligned} L\bar{\phi} &= G\phi', \\ L\phi' &= G\bar{\phi}, \end{aligned}\right\} \qquad (3.2\,a, b)$$

where only the particular integral of the second equation, generated by direct scattering out of the mean field, is required. The initial conditions on $\bar{\phi}$ are unchanged, of course. In principle this pair of equations may be solved as soon as the Green's function operator L^{-1} has been determined. However, this is quite often difficult, if not impossible, to obtain in a convenient and usable form. Actually, in the following we shall see that the need for L^{-1} to be known with any degree of precision is quite illusory. Indeed, use of the exact Green's function often leads to complications in the analysis which are not warranted by the amount of extra information so derived.

From a physical point of view, we expect a wave packet $\bar{\phi}$ propagating according to (3.1) to have the following properties which distinguish it from a similar wave packet propagating in the homogeneous medium, and therefore satisfying equation (2.1):

(i) The velocity of propagation of the wave packet will be reduced, essentially because it effectively spends a greater time in covering a given distance, owing to the 'buffeting' it receives from the random inhomogeneities.

(ii) Because of the scattering of the mean field energy and the consequent permanent creation of a random wave packet 'gas', the energy of the mean field is gradually *absorbed* by the medium.

Both effects (i) and (ii) turn out to be second order in the random fluctuations. However, (ii) is quite clearly of considerably more interest than (i), which only produces a small change in the undisturbed propagation velocity, whereas (ii) describes the physically most important *macroscopic* effect of the random medium. In § 4 we shall take the view that the most important piece of information to be derived from the theory is the size of this damping effect. This leads to a remarkable simplification of the calculations involved. In terms of a visco-elastic description of the medium this approximation corresponds to a neglect of elasticity.

In order to examine a simple, yet still rather general, class of wave-propagation problems, we shall suppose that the random operator G may be factorized into the following form:

$$G = G_1\left(\frac{\partial}{\partial x_i}\right)\left\{\xi(\mathbf{x})G_2\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial t}\right)\right\}, \tag{3.3}$$

where $\xi(\mathbf{x})$ is a random function of position $\mathbf{x}$ in the medium characterizing the randomness of the operator G. The time is denoted by t. The assumption that ξ is independent of the time covers a wide range of problems. Cases where this is manifestly not so (e.g. when ξ represents a turbulent fluctuation) are still well approximated by this assumption, provided that the time of passage of the incident wave packet $\bar{\phi}$ is small compared to the time scale of the fluctuations. The operators G_1, G_2 may be regarded as non-random polynomial functions of the derivatives in space and time, as indicated; G_1 operates on everything appearing on its right, i.e. including the fluctuations ξ.

Thus, in terms of the assumed form (3.3), the binary collision scattering equations (3.2) take the form,

$$L\bar{\phi} = G_1\left(\frac{\partial}{\partial x_i}\right)\overline{\left\{\xi(\mathbf{x})G_2\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial t}\right)\right\}\phi'}, \tag{3.4}$$

and

$$L\phi' = G_1\left(\frac{\partial}{\partial x_i}\right)\left\{\xi(\mathbf{x})G_2\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial t}\right)\right\}\bar{\phi}. \tag{3.5}$$

At this stage, it is necessary to make an assumption regarding the form of the linear wave operator L. We shall assume it to be a real, linear *differential* operator,

$$L = L\left(\frac{\partial}{\partial \mathbf{x}}, \frac{\partial}{\partial t}\right).$$

The first step in the analysis is to determine the particular integral of (3.5) in terms of $\bar{\phi}$. This must, of course, satisfy the radiation or causality condition. To do this we use the method of Fourier transformation. The Fourier transform $f(\mathbf{k}, \omega)$ of a function $f(\mathbf{x}, t)$ is defined by

$$f(\mathbf{k}, \omega) = \frac{1}{(2\pi)^{n+1}}\int\int_{-\infty}^{\infty} f(\mathbf{x}, t)\exp\left[-i(\mathbf{k}.\mathbf{x}-\omega t)\right]d\mathbf{x}\,dt, \tag{3.6a}$$

with

$$f(\mathbf{x}, t) = \int\int_{-\infty}^{\infty} f(\mathbf{k}, \omega)\exp\left[i(\mathbf{k}.\mathbf{x}-\omega t)\right]d\mathbf{k}\,d\omega, \tag{3.6b}$$

where n is equal to the number of space dimensions involved.

Actually, the definitions (3.6) are purely formal, and in general certain restrictions on the form of the function $f(\mathbf{x}, t)$, as well as on the region of integration in complex $(\mathbf{k}, \omega)$-space, must be imposed. In a causal system of the present type, a wave-function $f(\mathbf{x}, t)$ will vanish for $t < t_0$, say, and outside a finite region of space. If we assume $f(\mathbf{x}, t)$ to have no more than an exponential growth rate for large time, i.e.

$$|f(\mathbf{x}, t)| \lesssim e^{\epsilon t}, \tag{3.7}$$

for some fixed $\epsilon > 0$, then convergence of the first of the integrals, (3.6*a*), is ensured provided that

$$\operatorname{Im}(\omega) > \epsilon.$$

From this it is concluded that, for each fixed wave-number vector $\mathbf{k}, f(\mathbf{k}, \omega)$ is a regular function of ω provided that $\mathrm{Im}\,(\omega) > \epsilon$. Further, for such a restriction on $\omega, f(\mathbf{k}, \omega)$ is also a regular function of $\mathbf{k}$ throughout complex $\mathbf{k}$-space.

Thus, in formal manipulations of Fourier integrals (3.6*a*), it will be assumed that the imaginary part of ω is large enough to ensure convergence. Similarly, the path of integration in the ω-plane for integrals (3.6*b*) must pass above all the singularities of $f(\mathbf{k}, \omega)$. This being the case, it follows immediately that the radiation or causality condition is satisfied by integrals such as (3.6*b*). Indeed, the integral will vanish for $t < t_0$, since for such t the path of integration in the ω-plane may be displaced to $+i\infty$ along which the integrand is exponentially small.

Poles of $f(\mathbf{k}, \omega)$ which lie in the upper complex ω-plane correspond to exponentially growing components of $f(\mathbf{x}, t)$, and, in general, represent instabilities in the linearized system under discussion. The assumption (3.7) essentially places a limit on the size of the maximum growth rate of the system. In the following, we shall assume that the appropriate paths of integration have been chosen to conform with the above conditions.

With these ideas in mind, we first take the Fourier transform of (3.5) with respect to wave-number $\mathbf{K}$ and frequency Ω:

$$L(i\mathbf{K}, -i\Omega)\phi'(\mathbf{K}, \Omega) = \frac{1}{(2\pi)^{n+1}} \int\int G_1(iK_i)\xi(\mathbf{X}) G_2\left(\frac{\partial}{\partial X_i}, \frac{\partial}{\partial T}\right) \times \bar{\phi}(\mathbf{X}, T) \exp\left[-i(\mathbf{K}.\mathbf{X} - \Omega T)\right] d\mathbf{X}\, dT.$$

Hence, dividing by $L(i\mathbf{K}, -i\Omega)$ and using the inverse transform,

$$\phi'(\mathbf{x}, t) = \frac{1}{(2\pi)^{n+1}} \int\int\int\int \frac{G_1(iK_i)\xi(\mathbf{X})G_2\left(\frac{\partial}{\partial X_i}, \frac{\partial}{\partial T}\right)\bar{\phi}(\mathbf{X}, T)}{L(i\mathbf{K}, -i\Omega)} \times \exp\left[i\{\mathbf{K}.(\mathbf{x}-\mathbf{X}) - \Omega(t-T)\}\right] d\mathbf{X}\, dT\, d\mathbf{K}\, d\Omega.$$

It follows that, if $\xi(\mathbf{X})$ is a stationary random function of position, and

$$R(\mathbf{x}-\mathbf{X}) = \overline{\xi(\mathbf{x})\xi(\mathbf{X})}/\overline{\xi^2}, \tag{3.8}$$

where $\overline{\xi^2} = \overline{\xi(\mathbf{x})^2}$, then the right-hand side of (3.4) may be expressed in the form,

$$G_1\left(\frac{\partial}{\partial x_j}\right)\overline{\left\{\xi(\mathbf{x})\, G_2\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial t}\right)\right\}\phi'(\mathbf{x}, t)}$$
$$= \frac{\overline{\xi^2}}{(2\pi)^{n+1}} \int\int\int\int \frac{G_1(iK_i)\, G_2(iK_j, -i\Omega) R(\mathbf{z})}{L(i\mathbf{K}, -i\Omega)} \times G_1\left(\frac{\partial}{\partial x_j}\right) G_2\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial t}\right) \bar{\phi}(\mathbf{x}-\mathbf{z}, t-\tau) \exp\left[i\{\mathbf{K}.\mathbf{z} - \Omega\tau\}\right] d\mathbf{z}\, d\tau\, d\mathbf{K}\, d\Omega. \tag{3.9}$$

The next step is to substitute this into (3.4) and to take the Fourier transform of the resulting equality. After dividing through by $\bar{\phi}(\mathbf{k}, \omega)$, this gives, finally,

$$L(i\mathbf{k}, -i\omega) = \overline{\xi^2} \int_{-\infty}^{\infty} \frac{H(\mathbf{k}, \mathbf{K}, \omega)\psi(\mathbf{k}-\mathbf{K})}{L(i\mathbf{K}, -i\omega)}\, d\mathbf{K}, \tag{3.10}$$

Im $(\omega) > \epsilon$, where

$$H(\mathbf{k}, \mathbf{K}, \omega) = G_1(ik_j) G_2(iK_j, -i\omega) G_1(iK_i) G_2(ik_i, -i\omega), \tag{3.11}$$

and $\psi(\mathbf{k})$ is the Fourier space transform of the correlation function $R(\mathbf{x})$. In general, (3.11) involves scalar products of $\mathbf{k}$ and $\mathbf{K}$, indicated by the repeated suffixes.

Equation (3.10), which is the main result of this section, is the *dispersion equation* of binary collision theory governing the propagation of mean field wave packets. The 'power spectrum' of the random fluctuations of the medium is essentially given by $\psi(\mathbf{k})$, which often takes its maximum value at, or in the region of, the origin. Hence it may be anticipated that a large contribution to the integral (3.10) will come from those wave-numbers $\mathbf{K}$ which lie in the vicinity of the mean field wave-number $\mathbf{k}$. This is simply an indication of the prominence of *forward scattering*, and is a dominant aspect of the scattering of high-frequency incident waves (cf. Lighthill 1953, §3).

4. The dispersion equation and scattering cross-section

Equation (3.10) is the *exact* dispersion equation of binary collision theory. However, its usefulness is limited by our ability to perform the integration over $\mathbf{K}$-space. In general this is only possible in exceptional circumstances. Actually, such an evaluation tends to be facilitated, when possible, if the wave-number integration in (3.9) is carried out first; this gives the Green's function, and one is left with an integration over $\mathbf{z}$, corresponding to a 'retarded potential' type of solution.

In an arbitrary medium it is not normally possible to determine the Green's function explicitly. Further, it often happens that even with a precise knowledge of the Green's function the expressions take on such a complicated form that the subsequent integration over $\mathbf{z}$ becomes intractable.

In the present section we show how valuable information regarding the *absorption* of mean field energy by the random medium may be derived without a precise knowledge of the integral on the right of (3.10) (cf. (ii), §3). We shall restrict the discussion to a consideration of *stable* systems without dissipation. This means that, for each fixed *real* wave-number vector $\mathbf{k}$, the roots of the homogeneous dispersion equation

$$L(i\mathbf{k}, -i\omega) = 0 \tag{4.1}$$

lie on the real ω-axis. Hence in the integrand on the right of (3.9) the singularities in the Ω-plane lie on the real axis, and so the integration over Ω may be performed along a line parallel to the real axis and displaced slightly into the upper half plane. It follows that in performing the integration over $\mathbf{K}$ in (3.10) ω must be regarded as having a small *positive* imaginary part which is afterwards allowed to tend to zero. It is also true that, since there is no dissipation in the wave system, except for the absorption of mean field energy by the random wave field, then $H(\mathbf{k}, \mathbf{K}, \omega)$ must be *real* for real $\mathbf{k}$, $\mathbf{K}$ and ω. This is a consequence of the fact that in a non-dissipative system the characteristics of the exact equations (2.1) and (2.2) must be real.

Now (3.10) is the dispersion equation of mean field wave packets as calculated on binary collision theory. But that theory is essentially on $O(\overline{\xi^2})$ correction to the homogeneous equation (2.1) due to the presence of random elements. Hence it is appropriate to use (3.10) to calculate the corresponding correction to the roots, ω, of the homogeneous dispersion equation (4.1). Thus suppose that

$$\omega = \omega_0 + \overline{\xi^2}\omega_1, \tag{4.2}$$

where ω_0 is a root of equation (4.1). Then

$$L(i\mathbf{k}, -i\omega) = L(i\mathbf{k}, -i\omega_0) + \overline{\xi^2}\omega_1 \left(\frac{\partial L}{\partial \omega}\right)_{\omega=\omega_0} = \omega_1 \overline{\xi^2}\left(\frac{\partial L}{\partial \omega}\right)_{\omega=\omega_0} \tag{4.3}$$

Hence, by (3.10),

$$\omega_1 = \frac{1}{(\partial L/\partial \omega)_{\omega=\omega_0}} \int_{-\infty}^{\infty} \frac{H(\mathbf{k}, \mathbf{K}, \omega_0)\,\psi(\mathbf{k}-\mathbf{K})}{L(i\mathbf{K}, -i\omega_0)}\, d\mathbf{K}. \tag{4.4}$$

But the integration in (4.4) is to be performed in the limit as $\operatorname{Im}(\omega_0) \to +0$, which is the same thing as setting $\operatorname{Im}(\omega_0)$ equal to zero and indenting the path of integration in **K**-space to pass around the poles of the integrand in the appropriate sense. On such a region of integration the integrand is real provided that **K** is real. On the indentations, where **K** becomes complex, the integrand will become complex and there will be a *complex* contribution, in fact a *purely imaginary* contribution, to the integral. The remaining integration over real values of **K** is equal to the Cauchy principal value of the integral (4.4), and is real.

In this way we see that the integral in (4.4) may be separated into real and imaginary parts, to give

$$\omega_1 = \frac{A(\mathbf{k}, \omega_0) + iB(\mathbf{k}, \omega_0)}{(\partial L/\partial \omega)_{\omega=\omega_0}}, \tag{4.5}$$

where A and B are real functions of **k** and ω_0.

The real part of ω_1 represents the modification of the frequency due to the random buffeting experienced by the mean field wave packet. The imaginary part (which in applications turns out to be negative) represents the damping of the mean field due to scattering off the random inhomogeneities.

Now the main difficulty associated with the evaluation of the integral in (4.4) lies in the calculation of the Cauchy principal value contribution. The contribution from the poles may be obtained by standard complex variable theory. However, as already mentioned, the real part of (4.5) merely serves to alter slightly the velocity of propagation, whereas the imaginary part gives rise to damping, which is absent in the homogeneous equation (2.1). In applications it is only the damping factor that is normally required, and we shall therefore neglect the small *real* correction to the frequency.

Hence, from (4.4),

$$\omega_1 = \frac{1}{(\partial L/\partial \omega)_{\omega=\omega_0}} \int d\nu \int_{0}^{\infty} \frac{K^{n-1} H(\mathbf{k}, \mathbf{K}, \omega_0)\,\psi(\mathbf{k}-\mathbf{K})}{L(i\mathbf{K}, -i\omega_0)}\, dK, \tag{4.6}$$

where $d\mathbf{K} = K^{n-1}\, dK\, d\nu$, and $d\nu$ represents the product of the differential elements constituting the solid angle in **K**-space. For example, in two-dimensions $d\nu = d\theta$, the polar angle, and in three dimensions $d\nu = \sin\theta\, d\theta\, d\phi$, in the usual notation of polar co-ordinates.

In terms of polar co-ordinates, it is convenient to regard $L(i\mathbf{K}, -i\omega_0)$ as a function of K, ω_0 and the angular variables specifying the direction of $\mathbf{K}$ and denoted by ν, and we shall write it in the form $L(K, \nu; \omega_0)$. For fixed ν and real ω_0, the poles of the integrand of (4.6) on the real K-axis lie at those real K given by

$$L(K, \nu; \omega_0) = 0.$$

When ω_0 has a small positive imaginary part $i\epsilon$ the root K is shifted by a small amount ΔK given by

$$L(K+\Delta K, \nu; \omega_0+i\epsilon) = 0,$$

i.e.

$$\Delta K \frac{\partial L}{\partial K} + i\epsilon \frac{\partial L}{\partial \omega} = 0,$$

or

$$\Delta K = -i\epsilon\left(\frac{\partial L}{\partial \omega}\Big/\frac{\partial L}{\partial K}\right). \tag{4.7}$$

The pole is displaced into the upper or lower complex K-plane, requiring respectively that the contour of integration be indented *below* or *above* the real K-axis, according as

$$\frac{\partial L}{\partial \omega}\Big/\frac{\partial L}{\partial K} \lessgtr 0. \tag{4.8}$$

Hence, the imaginary contribution to (4.6) is given by

$$\operatorname{Im}(\omega_1) = \sum_m \frac{-\pi}{(\partial L/\partial \omega)_0} \int \frac{\operatorname{sgn}(\partial L/\partial \omega)_m K_m^{n-1} \psi(\mathbf{k}-K_m \boldsymbol{\zeta})}{|\partial L/\partial K|_m} H(\mathbf{k}, K_m \boldsymbol{\zeta}, \omega_0)\, d\nu, \tag{4.9}$$

where the summation is over all real positive K_m satisfying $L(K_m, \nu; \omega_0) = 0$, $\boldsymbol{\zeta}$ is a unit vector from the origin in the direction of the solid angle element $d\nu$. Also

$$\left(\frac{\partial L}{\partial \omega}\right)_0 = \left(\frac{\partial L}{\partial \omega}(\mathbf{k}, \omega)\right)_{\omega=\omega_0} \quad \left(\frac{\partial L}{\partial \omega}\right)_m = \left(\frac{\partial L}{\partial \omega}(K_m, \nu; \omega)\right)_{\omega=\omega_0}$$

$$\left(\frac{\partial L}{\partial K}\right)_m = \left(\frac{\partial L}{\partial K}(K, \nu; \omega_0)\right)_{K=K_m}$$

Now in general we deal with systems in which energy is conserved. This means that at any instant the total amount of wave energy is fixed, so that the growth of the random wave field *necessarily* requires the decay of the coherent field. It can be seen that under certain conditions $\operatorname{Im}(\omega_1)$ is definitely negative, corresponding to damping of the mean field (cf. Fourier representation of wave-functions (3.6*b*)). In the important case in which $L(i\mathbf{k}, -i\omega)$ is a function of $|\mathbf{k}|$ and ω alone, and when further, for each fixed ω, all wave-number vectors satisfying $L(i\mathbf{k}, -i\omega) = 0$ have the same *length*, then

$$\operatorname{Im}(\omega_1) = \frac{-\pi k^{n-1}}{|(\partial L/\partial \omega)_0 (\partial L/\partial k)_0|} \int \psi(\mathbf{k}-k\boldsymbol{\zeta}) H(\mathbf{k}, k\boldsymbol{\zeta}, \omega_0)\, d\nu. \tag{4.10}$$

But $\psi(\mathbf{k}-k\boldsymbol{\zeta})$ is the Fourier transform of the correlation function of a stationary random process and is therefore non-negative. Also, we can generally assert that $H(\mathbf{k}, k\boldsymbol{\zeta}, \omega_0)$ is *positive*. Indeed in the present case $L(\partial/\partial \mathbf{x}, \partial/\partial t)$ may be regarded as a polynomial in ∇^2, $\partial^2/\partial t^2$, and since, in general, the random operator G is derived from L by allowing a physical parameter of the medium to become a random function of position, it is clear that a large number of physically relevant cases are coverèd when the operators G_1 and G_2 have the forms,

$$\left.\begin{aligned} G_1 &= P_1(\nabla^2)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\ldots, \\ G_2 &= P_2\left(\nabla^2, \frac{\partial^2}{\partial t^2}\right)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\ldots, \end{aligned}\right\} \tag{4.11}$$

where P_1, P_2 are polynomial expressions, with the convention that repeated suffixes in the product $G_1\xi G_2$ implies summation (cf. definition 3.3). It follows that $H(\mathbf{k}, k\boldsymbol{\zeta}, \omega)$ is positive, since, although it may possibly contain negative scalar products $\mathbf{k}.\boldsymbol{\zeta}$, these will occur in pairs multiplied together. Hence, the integral is positive definite, and so Im (ω_1) is negative definite.

We have thus seen that absorption of mean field energy comes from those wave-number vectors $\mathbf{K}$ of the integrand in (4.6) at which $L(i\mathbf{K}, -i\omega_0)$ vanishes. Now, for each fixed real $\mathbf{K}$, the roots ω of (4.1) furnish the totality of species of wave which can be supported by the medium. Conversely, if ω is a fixed real quantity, then only those waves with wave-number vector $\mathbf{K}$ satisfying (4.1) are propagated at frequency ω. Hence, if the incident mean field has frequency ω, then, since the medium is time independent, all scattered waves must have that frequency, and all possible scattered waves must therefore have wave-number vectors $\mathbf{K}$ satisfying (4.1). The integration in (4.6) illustrates this fact. Non-propagating wave-numbers (not satisfying (4.1)) are involved in the Cauchy principal value contribution to the integral, and merely serve to alter slightly the velocity of propagation of the wave. Propagating wave-numbers, however, give complex contributions to the integral, and correspond to absorption of mean field energy by the random wave field.

We conclude this section by illustrating how the above results may be used to calculate the *total* scattering cross-section per unit volume of medium. Consider, in fact, a steady-state situation in which the energy density of the mean field of wave-number $\mathbf{k}$ and frequency ω is equal to $\mathrm{E}(\mathbf{x})$. Let $\mathbf{v}(\mathbf{k})$ denote the *group velocity* of the incident mean wave field, and which therefore characterizes the velocity at which mean field energy propagates. Then the rate of absorption of mean field energy per unit volume of medium is given by

$$-\operatorname{div}\{\mathrm{E}(\mathbf{x})\,\mathbf{v}(\mathbf{k})\}.$$

Since the flux of energy is just equal to $\mathrm{E}(\mathbf{x})|\mathbf{v}(\mathbf{k})|$, it follows that the total scattering cross-section per unit volume of the random medium, σ, is given by

$$\sigma = \frac{-\operatorname{div}\{\mathrm{E}(\mathbf{x})\,\mathbf{v}(\mathbf{k})\}}{\mathrm{E}(\mathbf{x})|\mathbf{v}(\mathbf{k})|}. \tag{4.12}$$

Now, for a plane wave of wave-number $\mathbf{k}$,

$$\mathrm{E}(\mathbf{x}) = \mathrm{E}_0\exp\left\{\frac{2\,\mathrm{Im}\,(\omega_1)\overline{\xi^2}\mathbf{x}.\mathbf{v}(\mathbf{k})}{|\mathbf{v}(\mathbf{k})|^2}\right\}, \tag{4.13}$$

where E_0 is a reference energy density. Hence,

$$\sigma = -\frac{2\overline{\xi^2}\,\mathrm{Im}\,(\omega_1)}{|\mathbf{v}(\mathbf{k})|},$$

i.e. since $\mathbf{v}(\mathbf{k}) = \partial\omega/\partial\mathbf{k}$, we have finally,

$$\sigma = -\frac{2\overline{\xi^2}\,\mathrm{Im}\,(\omega_1)}{|\partial\omega/\partial\mathbf{k}|}. \tag{4.14}$$

5. Application to shallow water waves

The theory of the previous sections is now illustrated by a consideration of the problem of the propagation of surface gravity waves over a stretch of water with a rough, or irregular, bed. Since a precise knowledge of the depth variations in such a situation cannot be assumed known, it seems appropriate to regard the fluctuations in depth as a random function of position. In fact, we shall suppose that the variations in depth may be regarded as a stationary random function of position.

The equations of shallow water wave theory are derived by Stoker (1957). If u_i denotes the mean horizontal fluid velocity, then in terms of horizontal coordinates x_i ($i = 1, 2$) the time t, and the surface elevation $\phi(\mathbf{x}, t)$, the equation of motion on shallow water theory has the form,

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -g\frac{\partial \phi}{\partial x_i}; \tag{5.1}$$

and the equation of continuity is

$$\frac{\partial}{\partial x_j}(hu_j) + \frac{\partial \phi}{\partial t} = -\frac{\partial}{\partial x_j}(\phi u_j), \tag{5.2}$$

provided that h, the undisturbed depth of the water, varies slowly on a scale of depth. This depth, $h(\mathbf{x})$, is a function of $\mathbf{x}$ alone. By differentiating (5.2) partially with respect to time, and substituting for $\partial u_j/\partial t$ from (5.1), we obtain the nonlinear shallow water equation,

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial}{\partial x_i}\left(gh\frac{\partial \phi}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(hu_j\frac{\partial u_i}{\partial x_j}\right) - \frac{\partial^2}{\partial t\,\partial x_i}(\phi u_i). \tag{5.3}$$

The terms on the right of this equation represent the effects of self-modulation of the wave form due to finite amplitude. These are neglected on linear theory (but see § 7). If we set

$$h = h_0\{1 + \xi(\mathbf{x})\}, \tag{5.4}$$

where h_0 represents the mean undisturbed depth and $\xi(\mathbf{x})$ is a stationary random function of $\mathbf{x}$ representing *small* fluctuations in the depth, then, with $a^2 = gh_0$, the linearized form of (5.3) becomes

$$\frac{\partial^2 \phi}{\partial t^2} - a^2\frac{\partial^2 \phi}{\partial x_i^2} = a^2\frac{\partial}{\partial x_i}\left(\xi(\mathbf{x})\frac{\partial \phi}{\partial x_i}\right). \tag{5.4}$$

Now the random operator on the right of this equation has precisely one of the forms included in definition (3.3). We may therefore apply the whole of the binary collision theory developed above to discuss the propagation of mean field surface wave packets. Further, the differential operators on the left of (5.4) constitute the operator $L(\partial/\partial\mathbf{x}, \partial/\partial t)$, and clearly $L(i\mathbf{k}, -i\omega)$ is a function of $|\mathbf{k}|$ and ω alone. Hence, we can use the formula (4.10) to obtain the following frequency correction:

$$\operatorname{Im}(\omega_1) = -\frac{\pi a k}{4}\int \psi(\mathbf{k}-k\boldsymbol{\zeta})(\mathbf{k}.\boldsymbol{\zeta})^2 d\nu. \tag{5.5}$$

6. The limits of short and long wavelengths

In the absence of further information concerning the form of the function $\psi(\mathbf{K})$, it is not possible to proceed further than the formula (5.5). However, in the limiting cases of short and long wavelengths it is possible to derive approximate expressions for the integral involved in terms of the 'integral scale' of the random fluctuations $\xi(\mathbf{x})$.

To do this we first let the angle between $\mathbf{k}$ and $\boldsymbol{\zeta}$ be denoted by θ, then, if we temporarily take $\mathbf{k}$ parallel to the x_1-axis, and $\boldsymbol{\zeta} = \boldsymbol{\zeta}(\theta) = (\cos\theta, \sin\theta)$, we have that

$$k\int \psi(\mathbf{k}-k\boldsymbol{\zeta})(\mathbf{k}.\boldsymbol{\zeta})^2 d\nu = k^3\int_0^{2\pi} \psi(\mathbf{k}-k\boldsymbol{\zeta})\cos^2\theta\, d\theta. \tag{6.1}$$

Suppose now that $\mathbf{K} = \mathbf{k}-k\boldsymbol{\zeta}(\theta)$. Then the integral in (6.1) is to be performed over a circle of radius k in $\mathbf{K}$-space shown in figure 1. In the limit of short wavelengths (k large), we shall suppose that k is much greater than the dominant wave-numbers of the 'power spectrum' $\psi(\mathbf{K})$. Then $\psi(\mathbf{K})$ is only significant when $|\mathbf{K}|/k$ is small. The extent of the region of wave-number space occupied by the power spectrum in relation to the circle of integration C is illustrated in figure 1 (cf. Lighthill (1953), who used this diagram in the study of sound scattering by turbulence). It is clear that the major contribution to the integral (6.1) comes from that part of the circle where K_1 is small, and this is approximately the part of the plane $K_1 = 0$ where $\psi(\mathbf{K})$ is significant. From this, and making the approximation $\cos\theta = 1$, it follows that the integral (6.1) becomes approximately

$$k^2\int_{-\infty}^{\infty} \psi(0, K_2)\, dK_2 = \frac{k^2}{2\pi}\int_{-\infty}^{\infty} R(x_1, 0)\, dx_1 = \frac{k^2 L}{\pi}, \tag{6.2}$$

where L is the integral scale of the random fluctuations in the $\mathbf{k}$-direction. Hence, we arrive at the short wavelength approximation to (5.5)

$$\operatorname{Im}(\omega_1) = -aLk^2/4. \tag{6.3}$$

Next, let us consider the opposite limit in which the wavelength of the incident mean field is *large* compared to those of the dominant harmonic components of $\psi(\mathbf{K})$. In this case, the circle of integration C in figure 1 will lie completely within the region of wave-number space occupied by $\psi(\mathbf{K})$. For sufficiently small $\mathbf{k}$ (i.e. long wavelength), it is therefore appropriate to expand $\psi(\mathbf{K})$ in a power series in $\mathbf{K}$. Suppose for simplicity that the first non-zero term in this expansion is $\psi(\mathbf{o})$, then (6.1) becomes

$$k^3\psi(\mathbf{o})\int_0^{2\pi} \cos^2\theta\, d\theta = \pi k^3\psi(\mathbf{o}). \tag{6.4}$$

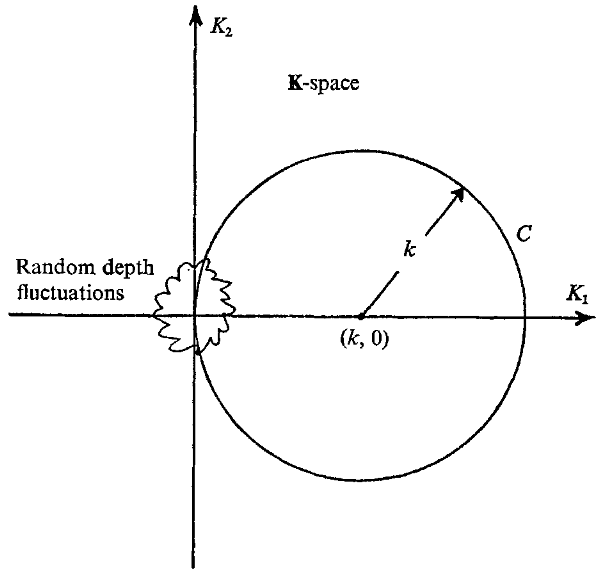

FIGURE 1. The circle of integration C in **K**-space in relation to the region of wave-number space occupied by the random depth fluctuations. The figure illustrates the case of a short wavelength incident wave packet.

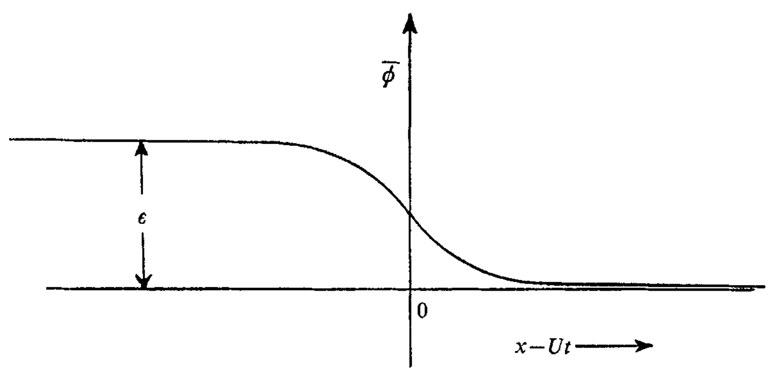

FIGURE 2. The mean profile of the shallow water bore.

But
$$\psi(\mathbf{o}) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} R(\mathbf{z})\,d\mathbf{z} = \frac{L^2}{\pi^2}, \tag{6.5}$$

where it has been assumed that the integral scale L is independent of direction. Hence, the long wavelength approximation becomes

$$\operatorname{Im}(\omega_1) = -ak^3L^2/4. \tag{6.6}$$

Actually, we may now use the limiting results (6.3) and (6.6) to derive approximate *macroscopic* equations for the mean field containing damping terms. First, consider the short wavelength case given by (6.3). Referring to the definition (4.2), and noting that in the case of shallow water waves $\omega_0 = \pm ak$, we obtain

$$\omega = \pm ak - \frac{iaLk^2}{4}\overline{\xi^2} \tag{6.7}$$

for the corrected eigenfrequency, including the effects of binary collision damping, but excluding the small effect of real frequency shift due to passage through the medium. But the frequencies (6.7) are the roots of the dispersion equation

$$(\omega + (iaL\overline{\xi^2}/4)k^2)^2 - a^2k^2 = 0,$$

i.e. correct to terms in $\overline{\xi^2}$,

$$\omega^2 - a^2k^2 + (iaL\overline{\xi^2}/2)\omega k^2 = 0. \tag{6.8}$$

Now, this is the dispersion equation of the equation,

$$\frac{\partial^2 \overline{\phi}}{\partial t^2} - a^2 \frac{\partial^2 \overline{\phi}}{\partial x_i^2} = \left(\frac{aL\overline{\xi^2}}{2}\right) \frac{\partial^3 \overline{\phi}}{\partial t\, \partial x_i^2}, \tag{6.9}$$

which therefore describes the propagation of the mean wave field in the short wavelength limit. The term on the right clearly has the interpretation of a viscous damping effect.

Similar reasoning may be applied in the limit of long wavelengths. Here

$$\omega = \pm ak - (iaL^2k^3/4)\,\overline{\xi^2}, \tag{6.10}$$

which gives the roots of the dispersion equation

$$\omega^2 - a^2k^2 + (iaL^2\overline{\xi^2}/2)\omega k^3 = 0. \tag{6.11}$$

Now in this equation $k^3 = |\mathbf{k}|^3$, and has no direct analogue as a differential operator in $\mathbf{x}$-space. To gain further insight into the nature of the real space equation corresponding to (6.11) let us suppose for simplicity that the mean wave field $\overline{\phi}\,(\mathbf{x}, t)$ is a function of time and one space dimension x alone. Then, if (6.11) be multiplied by $\overline{\phi}(\mathbf{k}, \omega)$ and the inverse Fourier transform of the resulting equation is taken, we obtain the following mean field wave equation:

$$\frac{\partial^2 \overline{\phi}}{\partial t^2} - a^2 \frac{\partial^2 \overline{\phi}}{\partial x^2} = \alpha^2 \frac{\partial^4}{\partial t\, \partial x^3} \int_{-\infty}^{\infty} \frac{\overline{\phi}(X, t)}{x - X}\, dX, \tag{6.12}$$

where the integral on the right is a Cauchy principal value integral, and

$$\alpha^2 = aL^2\overline{\xi^2}/2\pi. \tag{6.13}$$

7. The non-linear shallow water wave equation

Equations (6.9) and (6.12) derived above are macroscopic equations, which describe the propagation of the mean wave field without specific reference to the presence of the random wave packet 'gas' responsible for the dissipation. In gas dynamics, dissipation enters through the viscous term in the equation of motion, which in fact is a macroscopic approximation to the average properties of the discrete molecular components of the gas. In the theory of weak shock waves (Lighthill 1956), it is customary to balance this macroscopic dissipation term against the non-linear convective term. In this way, one is able to compute, for example, the steady state form of a weak shock wave.

The possibility arises, therefore, of applying the theory presented in the earlier sections of this paper to the study of such non-linear phenomena. In particular, it would be of great interest to be able to effect a balance between the macroscopic dissipation term due to random scattering and any non-linear terms appearing in the full equation in the hope of obtaining steady state 'shock' profiles. The idea of applying the present method to such problems was originally suggested to the author by Professor M. J. Lighthill, and it has already been successfully applied to the study of the propagation of weak shock waves through atmospheric turbulence (Ffowcs Williams & Howe 1971).

Let us now outline the procedure by considering the problem of the shallow water bore, or hydraulic jump. In strong bores the main mechanism of energy loss is due to turbulent dissipation and churning-up of the flow (Lighthill 1957), but weaker bores have a different structure. Behind a sufficiently weak bore there is a tendency for a train of stationary waves to form which exhibit no breaking, the flow appearing perfectly smooth. Energy dissipation then takes place principally by radiation through this stationary wave train. It may be argued, however, that the presence of irregularities in the river bed would introduce an additional means of dissipation due to wave scattering. For weak bores such damping could conceivably account for all of the required dissipation at the bore, and so inhibit the formation of the stationary wave train in its wake. We shall take the view, therefore, that the steady state of such a weak bore is maintained solely by means of a balance between non-linear convective terms and the dissipation due to wave scattering off the random irregularities of the river bed.

We start by deriving a binary collision theory for the non-linear shallow water wave equation (5.3) This may be expressed in the form:

$$\frac{\partial^2\phi}{\partial t^2}-a^2\frac{\partial^2\phi}{\partial x_i^2}=a^2\frac{\partial}{\partial x_i}\left(\xi(\mathbf{x})\frac{\partial\phi}{\partial x_i}\right)+\frac{\partial}{\partial x_i}\left(hu_j\frac{\partial u_i}{\partial x_j}\right)-\frac{\partial^2}{\partial t\,\partial x_i}(\phi u_i). \tag{7.1}$$

Assume that for a theory of *weak* bores it is sufficient to retain terms up to and including second order. This means that (7.1) may be approximated by

$$\frac{\partial^2\phi}{\partial t^2}-a^2\frac{\partial^2\phi}{\partial x_i^2}=a^2\frac{\partial}{\partial x_i}\left(\xi(\mathbf{x})\frac{\partial\phi}{\partial x_i}\right)+h_0\frac{\partial}{\partial x_i}\left(u_j\frac{\partial u_i}{\partial x_j}\right)-\frac{\partial^2}{\partial t\,\partial x_i}(\phi u_i). \tag{7.2}$$

Now take the ensemble average of this equation in the manner described in §2. Setting in the usual notation,

$$\left.\begin{aligned}\phi&=\overline{\phi}+\phi',\\ u_i&=\overline{u}_i+u_i',\end{aligned}\right\} \tag{7.3}$$

we obtain

$$\begin{aligned}\frac{\partial^2\overline{\phi}}{\partial t^2}-a^2\frac{\partial^2\overline{\phi}}{\partial x_i^2}&=a^2\frac{\partial}{\partial x_i}\overline{\left(\xi(\mathbf{x})\frac{\partial\phi'}{\partial x_i}\right)}+h_0\frac{\partial}{\partial x_i}\left(\overline{u}_j\frac{\partial\overline{u}_i}{\partial x_j}\right)\\ &\quad+h_0\frac{\partial}{\partial x_i}\overline{\left(u_j'\frac{\partial u_i'}{\partial x_j}\right)}-\frac{\partial^2}{\partial t\,\partial x_i}(\overline{\phi}.\overline{u}_i)-\frac{\partial^2}{\partial t\,\partial x_i}(\overline{\phi' u_i'}).\end{aligned} \tag{7.4}$$

Subtracting this from (7.2), we obtain the equation for ϕ':

$$\begin{aligned}\frac{\partial^2\phi'}{\partial t^2}-a^2\frac{\partial^2\phi'}{\partial x_i^2}&=a^2\frac{\partial}{\partial x_i}\left(\xi(\mathbf{x})\frac{\partial\overline{\phi}}{\partial x_i}\right)+a^2\left\{\frac{\partial}{\partial x_i}\left(\xi(\mathbf{x})\frac{\partial\phi'}{\partial x_i}\right)-\frac{\partial}{\partial x_i}\overline{\left(\xi(\mathbf{x})\frac{\partial\phi'}{\partial x_i}\right)}\right\}\\ &\quad+h_0\frac{\partial}{\partial x_i}\left\{u_j'\frac{\partial\overline{u}_i}{\partial x_j}+\overline{u}_j\frac{\partial u_i'}{\partial x_j}\right\}+h_0\frac{\partial}{\partial x_i}\left\{u_j'\frac{\partial u_i'}{\partial x_j}-\overline{u_j'\frac{\partial u_i'}{\partial x_j}}\right\}\\ &\quad-\frac{\partial^2}{\partial t\,\partial x_i}\{\phi'\overline{u}_i+\overline{\phi}u_i'\}-\frac{\partial^2}{\partial t\,\partial x_i}\{\phi'u_i'-\overline{\phi'u_i'}\}.\end{aligned} \tag{7.5}$$

The usual approximation of binary collision theory then permits simplification of this to

$$\frac{\partial^2\phi'}{\partial t^2}-a^2\frac{\partial^2\phi'}{\partial x_i^2}=a^2\frac{\partial}{\partial x_i}\left(\xi(\mathbf{x})\frac{\partial\overline{\phi}}{\partial x_i}\right)+h_0\frac{\partial}{\partial x_i}\left(u_j'\frac{\partial\overline{u}_i}{\partial x_j}+\overline{u}_j\frac{\partial u_i'}{\partial x_j}\right)-\frac{\partial^2}{\partial t\,\partial x_i}(\phi'\overline{u}_i+\overline{\phi}u_i'). \quad (7.6)$$

Now the second pair of terms on the right of (7.6) are of order $|\xi|\overline{\phi}^2$, so that the component of ϕ' corresponding to these terms will give rise to terms non-linear in the mean field and of order $|\xi|^2\overline{\phi}^2$ when substituted into (7.4). Such non-linear terms will be neglected in comparison with those of order $\overline{\phi}^2$. Similarly, only the mean field non-linear terms need be retained in (7.4). Hence, we arrive at the following pair of binary collision equations:

$$\left.\begin{aligned}\frac{\partial^2\overline{\phi}}{\partial t^2}-a^2\frac{\partial^2\overline{\phi}}{\partial x_i^2}&=a^2\frac{\partial}{\partial x_i}\overline{\left(\xi(\mathbf{x})\frac{\partial\phi'}{\partial x_i}\right)}+h_0\frac{\partial}{\partial x_i}\left(\overline{u}_j\frac{\partial\overline{u}_i}{\partial x_j}\right)-\frac{\partial^2}{\partial t\,\partial x_i}(\overline{\phi}\overline{u}_i),\\ \frac{\partial^2\phi'}{\partial t^2}-a^2\frac{\partial^2\phi'}{\partial x_i^2}&=a^2\frac{\partial}{\partial x_i}\left(\xi(\mathbf{x})\frac{\partial\overline{\phi}}{\partial x_i}\right).\end{aligned}\right\} \quad (7.7)$$

The second of (7.7) is precisely the random equation obtained on the linearized theory of the earlier sections (although not set down explicitly). Hence, it may be solved in precisely the formal manner discussed in §§3, 4, and the solution substituted into the mean field equation of (7.7). Without labouring on the mathematical details, let us adopt the notation of (3.1), and write the mean field equation in the form,

$$\frac{\partial^2\overline{\phi}}{\partial t^2}-a^2\frac{\partial^2\overline{\phi}}{\partial x_i^2}=\overline{GL^{-1}G}\,\overline{\phi}+h_0\frac{\partial}{\partial x_i}\left(\overline{u}_j\frac{\partial\overline{u}_i}{\partial x_j}\right)-\frac{\partial^2}{\partial t\,\partial x_i}(\overline{\phi}\overline{u}_i). \quad (7.8)$$

We now seek steady solutions of this equation of the form,

$$\left.\begin{aligned}\overline{\phi}&=\overline{\phi}(x-Ut),\\ \overline{u}&=\overline{u}(x-Ut),\end{aligned}\right\} \quad (7.9)$$

where $\overline{\phi},\overline{u}$ depend only on one space co-ordinate x, U is a *constant* wave speed, and $\overline{u}$ is the x-direction mean velocity (the other component having zero mean). In particular, we shall look for a 'shock-like' solution of the type illustrated in figure 2. Under these circumstances, (7.8) reduces to

$$U^2\frac{\partial^2\overline{\phi}}{\partial x^2}-a^2\frac{\partial^2\overline{\phi}}{\partial x^2}=\overline{GL^{-1}G}\,\overline{\phi}+\frac{h_0}{2}\frac{\partial^2\overline{u}^2}{\partial x^2}+\frac{U\partial^2}{\partial x^2}(\overline{\phi}\overline{u}). \quad (7.10)$$

To express (7.10) in terms of the unknown $\overline{\phi}$ alone, we make use of the averaged form of the continuity equation (5.2). Actually, it is permissible to use the linearized form of (5.2), since only second-order terms are being retained on the right of (7.10):

$$h_0\frac{\partial\overline{u}}{\partial x}+\frac{\partial\overline{\phi}}{\partial t}=0, \quad (7.11)$$

i.e. by (7.9),

$$\overline{u}=\frac{U}{h_0}\overline{\phi}. \quad (7.12)$$

Thus, (7.10) becomes

$$(U^2-a^2)\frac{\partial^2\overline{\phi}}{\partial x^2} = \overline{GL^{-1}G}.\overline{\phi}+\frac{3U^2}{2h_0}\frac{\partial^2}{\partial x^2}(\overline{\phi}^2). \tag{7.13}$$

It remains to determine the form of $\overline{GL^{-1}G}.\overline{\phi}$. This term has already been seen to simplify in the limits of short and long waves on linear theory. First, consider the short wavelength limit.

Short wavelength limit

This requires that all wave-numbers, k, associated with $\overline{\phi}$ must satisfy

$$kL \gg 1,$$

where L is the correlation length of the random fluctuations.

Strictly, the following sequence of operations should be carried out to determine the form of $\overline{GL^{-1}G}\,\overline{\phi}$:

(*a*) Take the Fourier transform of equation (7.13).

(*b*) Substitute an approximate expression for the transform of $\overline{GL^{-1}G}\,\overline{\phi}$, bearing in mind that for waves travelling in the positive x-direction ω_0 and k must have the same sign.

(*c*) Take the inverse transform to give the appropriate approximate real space equation (7.13).

The procedure is straightforward and, as might have been anticipated, gives the same result as is obtained from the approximate form on the right of (6.9), provided that the operator $\partial/\partial t$ is replaced by $-a\partial/\partial x$ (the error introduced by this approximation is negligible, since the dissipation term is already of order $\overline{\xi^2}\overline{\phi}$). In this way, we obtain the following short wavelength approximation to (7.13):

$$(U^2-a^2)\frac{\partial^2\overline{\phi}}{\partial x^2} = -\frac{a^2L\overline{\xi^2}}{2}\frac{\partial^3\overline{\phi}}{\partial x^3}+\frac{3U^2}{2h_0}\frac{\partial^2}{\partial x^2}(\overline{\phi}^2). \tag{7.14}$$

Since, for a wave form of the type shown in figure 2, all derivatives vanish at $x = \pm\infty$, this reduces to

$$(U^2-a^2)\overline{\phi} = -\frac{a^2L\overline{\xi^2}}{2}\frac{\partial\overline{\phi}}{\partial x}+\frac{3U^2}{2h_0}\overline{\phi}^2. \tag{7.15}$$

Next suppose that the jump in depth across the bore is equal to ϵ, then, as $x\rightarrow-\infty$, (7.15) gives

$$U^2-a^2 = \frac{3U^2}{2h_0}\epsilon,$$

i.e.

$$U = \frac{a}{(1-[3\epsilon/2h_0])^{\frac{1}{2}}} > a, \tag{7.16}$$

which shows that the propagation velocity of the bore exceeds the shallow water wave speed by approximately

$$3a\epsilon/4h_0.$$

Using (7.16), the wave equation may be written

$$a^2L\overline{\xi^2}\frac{\partial\overline{\phi}}{\partial x} = \frac{3U^2}{h_0}\overline{\phi}(\overline{\phi}-\epsilon). \tag{7.17}$$

Hence we derive the form of the bore profile,

$$\bar{\phi} = \epsilon \Big/ 1+\exp\left\{\frac{3U^2\epsilon(x-Ut)}{a^2h_0L\bar{\xi^2}}\right\}$$

$$\simeq \epsilon \Big/ 1+\exp\left\{\frac{3\epsilon}{\bar{\xi^2}h_0L}(x-Ut)\right\}. \tag{7.18}$$

This should be compared with the theory presented by Lighthill (1956, p. 287) in his survey of the Taylor theory of weak shock waves. Following Lighthill we define the bore 'thickness' δ to be equal to the distance over which the surface elevation falls from 0.95ϵ to 0.05ϵ. This is readily computed from the above formula to be

$$\delta = \frac{h_0L\bar{\xi^2}}{3\epsilon}(2\log_e 19),$$

i.e.

$$\delta = 2\,h_0L\bar{\xi^2}/\epsilon. \tag{7.19}$$

Actually, it is instructive to derive the result (7.19) in an alternative manner. This depends on deciding at the outset to develop a polynomial approximation to the exact profile of the bore. To do this, we first define the dimensionless variable,

$$\eta = \frac{3U^2\epsilon}{a^2h_0L\bar{\xi^2}}(x-Ut) \simeq \frac{3\epsilon}{h_0L\bar{\xi^2}}(x-Ut),$$

which enables (7.17) to be written in the form,

$$\epsilon d\bar{\phi}/\partial\eta = \bar{\phi}(\bar{\phi}-\epsilon). \tag{7.20}$$

Referring to figure 2, it would appear to be reasonable to adopt a *cubic* approximation to the bore profile of the form,

$$\bar{\phi} = \epsilon\{\tfrac{1}{2}+\alpha\eta+\beta\eta^3\}, \tag{7.21}$$

valid over the range $(-l < \eta < l)$ of the profile. Then the bore thickness is equal to $2l$, and at

$$\eta = l, \quad \bar{\phi} = 0, \quad \text{and at} \quad \eta = -l, \quad \bar{\phi} = \epsilon.$$

In either case, we must have

$$\tfrac{1}{2} = -\alpha l - \beta l^3. \tag{7.22}$$

To ensure that the profile is 'smooth' we next require that the gradient $\partial\phi/\partial\eta$ vanish at $\eta = \pm l$. Hence

$$0 = \alpha + 3\beta l^2, \tag{7.23}$$

so that

$$\alpha = -3/4l \quad \text{and} \quad \beta = 1/4l^3,$$

and therefore

$$\bar{\phi} = \epsilon\left\{\frac{1}{2}-\frac{3\eta}{4l}+\frac{\eta^3}{4l^3}\right\}. \tag{7.24}$$

In deriving this approximation, we have used two conditions at $\eta = \pm l$, and one condition at $\eta = 0$, viz. that $\bar{\phi}(0) = \epsilon/2$. We complete the specification of (7.24) by satisfying (7.20) in the limit as $\eta \to 0$. This means that the approximation is two-ended, making use of two conditions at both ends of the range $(0, \pm l)$, and should therefore be more efficient than a Taylor series expansion from one end of the range. Thus, substituting (7.24) into both sides of equation (7.20), and setting $\eta = 0$, gives $$3/4l = \tfrac{1}{4},$$

i.e. $$l = 3,$$

and so
$$\overline{\phi} = \epsilon\left\{\frac{1}{2} - \frac{\eta}{4} + \frac{\eta^3}{108}\right\} \tag{7.25}$$

is the required approximation to the bore profile.

The dimensionless bore thickness $2l = 6$. Hence, the actual thickness δ is given by
$$\frac{\delta \,.\, 3\epsilon}{h_0 L\overline{\xi^2}} = 6, \tag{7.26}$$

i.e. $$\delta = 2h_0 L\overline{\xi^2}/\epsilon,$$

which is precisely the result (7.19).

We shall see that this approximate method is useful in determining the thickness in the long wavelength approximation considered below. Before turning to a consideration of this case it is of interest to point out that an approximate analysis of non-steady, non-linear wave forms can also be treated by a variant of the theory of this section. In this case one can derive the unsteady analogue of (7.17), viz.
$$\frac{\partial\overline{\phi}}{\partial t} + \frac{3a}{2h_0}\overline{\phi}\frac{\partial\overline{\phi}}{\partial X} = \frac{aL\overline{\xi^2}}{4}\frac{\partial^2\overline{\phi}}{\partial X^2}, \tag{7.27}$$

where $X = x - at$. But this is just Burger's equation, which has been discussed in great detail by Lighthill (1956). We shall not pursue this matter further here, but refer the interested reader to the original reference for further details.

Long wavelength limit

By reasoning similar to that indicated above it is readily shown that in the long wavelength limit the equation for the bore profile corresponding to (7.15) has the form,
$$(U^2 - a^2)\overline{\phi} = -\frac{a^2L^2\overline{\xi^2}}{2\pi}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty}\frac{\overline{\phi}(X)\,dX}{x-X} + \frac{3U^2}{2h_0}\overline{\phi}^2. \tag{7.28}$$

As before, $$U = \frac{a}{(1-[3\epsilon/2h_0])^{\frac{1}{2}}} \simeq a\left(1+\frac{3\epsilon}{4h_0}\right);$$

and (7.28) then simplifies to
$$\frac{h_0L^2\overline{\xi^2}}{3\pi}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty}\frac{\overline{\phi}(X)\,dX}{x-X} = \overline{\phi}(\overline{\phi}-\epsilon). \tag{7.29}$$

By means of the 'Poincare–Bertrand' lemma,
$$\int_{-\infty}^{\infty}\frac{d\lambda}{x-\lambda}\int_{-\infty}^{\infty}\frac{\partial^2\overline{\phi}}{\partial X^2}\frac{dX}{\lambda - X} = -\pi^2\frac{\partial^2\overline{\phi}}{\partial x^2},$$

we finally transform (7.29) into
$$\frac{\pi h_0L^2\overline{\xi^2}}{3}\frac{\partial^2\overline{\phi}}{\partial x^2} = \int_{-\infty}^{\infty}\frac{\overline{\phi}(\epsilon-\overline{\phi})}{x-X}\,dX. \tag{7.30}$$

It is now clear by inspection that a solution of this non-linear integral equation of the form illustrated in figure 2 must exist.

Changing to the dimensionless variable,

$$\eta = (3\epsilon/\pi h_0 L^2 \overline{\xi^2})^{\frac{1}{2}} x,$$

(7.30) becomes

$$\epsilon \frac{\partial^2 \overline{\phi}}{\partial \eta^2} = \int_{-\infty}^{\infty} \frac{\overline{\phi}(\epsilon - \overline{\phi})}{\eta - \mu} d\mu. \tag{7.31}$$

Next, we adopt the cubic approximation (7.24) to the solution of (7.31) in the vicinity of the wave front. Then by substituting for $\overline{\phi}$ in (7.31), noting that the limits of integration now become $(-l, l)$, and equating the lowest order terms appearing on each side of the equation, we deduce that

$$l = \sqrt{\tfrac{15}{14}}. \tag{7.32}$$

Hence, the dimensionless bore thickness is equal to $2\sqrt{\frac{15}{14}}$, so that the actual thickness δ is given by

$$\delta = 2L(5\pi h_0 \overline{\xi^2}/14\epsilon)^{\frac{1}{2}}. \tag{7.33}$$

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